A positive conservative method for magnetohydrodynamics based on HLL and Roe methods

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Abstract

The exact Riemann problem solutions of the usual equations of ideal magnetohydrodynamics (MHD) can have negative pressures, if the initial data has $\nabla \cdot \mathbf{B} \neq 0$. This creates a problem for numerical solving because in a first order finite-volume conservative Godunov-type method one cannot avoid jumps in the normal magnetic field component even if the magnetic field was divergenceless in the three-dimensional sense. We show that by allowing magnetic monopoles in MHD equations and properly taking into account the magnetostatic contribution to the Lorentz force, an additional source term appears in Faraday's law only. Using the Harten-Lax-vanLeer (HLL) Riemann solver and discretizing the source term in a specific manner, we obtain a method which is positive and conservative. We show positivity by extensive numerical experimentation. This MHD-HLL method is positive and conservative but rather diffusive, thus we show how to hybridize this method with the Roe method to obtain a much higher accuracy while still retaining positivity. The result is a fully robust positive conservative scheme for ideal MHD, whose accuracy and efficiency properties are similar to the first order Roe method and which keeps $\nabla \cdot \mathbf{B}$ small in the same sense as Powell's method. As a special case, a method with similar characteristics for accuracy and robustness is obtained for the Euler equations as well.

1 Introduction

The equations of ideal magnetohydrodynamics (MHD) are widely used in many large-scale plasma physical simulations, and their importance is growing as high performance computing is becoming a "desktop" tool. As an example, there are plans to start using a three-dimensional MHD-based magnetosphere-ionosphere coupling simulation for operative space weather forecasting purposes. For such applications especially, the robustness of the code is very important since the program is expected at least to converge for any solar wind input, if not necessarily to produce a correct forecast in every situation.

An optimal scheme for the MHD equations would have the following four properties:

- Exact conservation of mass, momentum and energy
- Preservation of $\nabla \cdot \mathbf{B} = 0$ as accurately as possible
- Positivity (pressure and density remain positive under all circumstances)
- As small numerical dissipation as possible

Any finite-volume method (FVM) which is based on computing the fluxes at cell interfaces is conservative so the first property is not difficult to satisfy [11]. The $\nabla \cdot \mathbf{B} \approx 0$ requirement is more difficult, but if a scheme does not preserve $\nabla \cdot \mathbf{B} = 0$ within roundoff error, it might preserve it within truncation error [15, 16], which is usually enough. In other case one has to periodically remove the divergence by a subtraction of the curl-free part, which is sometimes called a projective method or elliptic cleaning [19, 20, 21, 12]. Finding a provably positive scheme which is at the same time conservative is, on the other hand, still an unsolved problem in MHD, although certain kinetic solvers [2] may actually have these properties, but they are much more diffusive than Roe-type methods (low-diffusion kinetic-type solvers have been developed for Euler equations [13], but are not known in MHD). Lacking such a scheme, one has had to fight the positivity problem by adding explicit diffusion, fiddling with the initial and boundary conditions or grid refinement, modifying the wavespeeds, or by trying to fix negative pressures once they arise by adding energy locally. The list of tricks is long and not exhaustive. These usually help in individual cases, but none of these cures the problem for once and for all. The last point, accuracy, is easiest to achieve with Roe-type methods, which are based on some local linearization of the Riemann problem [11]. It is well known that Roe-type methods are not positive, which means that no linearization is able to produce a positive solution for all Riemann problems [4].

Our practical experience with magnetospheric MHD simulations [8, 9, 10] has shown that the positivity problem is worse and occurs more often than what one could assume after reading the literature, and that problems arise more often when one moves towards more realistic models with real solar wind input, automatically adaptive spatial and temporal grids and realistic ionospheric models. According to our experience, the most workable solution to cure negative pressures after they arise is simply to cancel the cell update step if its pressure would become negative. This method is sometimes able to fix negative pressures in a stable manner, although it is of course unphysical and breaks the conservation laws.

The connection between the pressure P, the total energy density U, the mass density ρ, the momentum flux $\mathbf{p} = \rho \mathbf{v}$, and magnetic field $\mathbf{B}$ is

$$P = (\gamma - 1)\left(U - \frac{p^2}{2\rho} - \frac{B^2}{2\mu_0}\right). \tag{1}$$

If a scheme is positive and conserves at least the total energy exactly, it is numerically stable in the sense that all energy components (thermal, kinetic and magnetic) remain bounded, if the flow through the boundaries is well behaved. The total energy $\int dVU$ remains bounded because of energy conservation. The kinetic and magnetic energy densities, and thus all components of the velocity and magnetic field, are also bounded because $P > 0$. Thus, a positive conservative scheme is fully robust and thus highly desirable.

By looking at (1) it is clear why it is not easy to develop conservative methods which are also positive. The pressure (thermal energy) is computed by subtracting the kinetic and magnetic energies from the total energy. Especially in a low-beta plasma ($\beta = 2\mu_0 P/B^2$), the total energy consists almost totally of magnetic energy, and thus subtracting the two is prone to all kinds of numerical errors which can easily turn P negative. The quantities U, ρ, $\mathbf{p}$ and $\mathbf{B}$ on the right-hand-side of (1) are themselves computed from their own conservation laws, which are seemingly unrelated to equation (1). Thus it is not clear *a priori* that conservative methods which are positive under all circumstances exist at all.

One can ease the positivity problems in a low-beta plasma somewhat if the strong magnetic field is a potential field. Then one can separate the background magnetic field analytically [20, 21]. We use this method in our global MHD simulation [8, 9, 10], but for simplicity we drop these considerations from this paper.

If one drops the conservation requirement and uses the primitive variable formulation or the semi-conservative formulation [17] where the pressure is one of the dynamical variables, it is easier to develop positive methods. However, when doing so one loses exact conservation which causes other shortcomings, such as possibly erroneous shock speeds [11].

It is intriguing that the Harten-Lax-vanLeer (HLL) scheme [6, 3, 4] is a positive scheme for any conservative hyperbolic system, whose exact Riemann problem solutions are positive for positive left and right states. In addition to the Euler equations, a planar Lagrangian MHD Riemann problem without a jump in the normal magnetic field has been shown to be well-posed [14]. The solution of a well-posed Riemann problem has non-negative pressure and density (zero pressure and density can occur, but this is usually not an issue for numerics). Thus the HLL method is conservative and positive for Euler equations, as was asserted by Einfeldt *et al.*[4].

A straightforward application of the HLL scheme to the seven-wave MHD equations reveals that the HLL middle state, which is an average of the exact Riemann problem solution, can become non-positive if there is a jump of the normal component of the magnetic field across the cell interface. This implies that even the exact solutions to MHD Riemann problems sometimes fail to be positive, if there is a jump in the cell interface normal component of the magnetic field B_x, i.e., a nonzero $\nabla \cdot \mathbf{B}$, in the initial data. If there is no jump in B_x, the HLL method for MHD seems to be positive if the bounds to the wavespeeds are large enough (we have found no counterexamples). This also follows from the well-posedness of MHD mentioned above.

In a two- or three-dimensional first-order Cartesian FVM which is based on solving one-dimensional Riemann problems, we cannot avoid jumps in B_x at cell interfaces even if $\nabla \cdot \mathbf{B} = 0$ is valid in a multidimensional sense. Thus, to find a positive conservative scheme for MHD we seem to have two possible routes. The first route is to abandon the first-order FVM, which is based on piecewise constant states, and go to higher order methods. Higher order representations of the magnetic field can be made divergence-free, as can a surface-averaged representation. The second route is to modify the MHD equations so that they give positive solutions even when $\nabla \cdot \mathbf{B} \neq 0$. In this paper we pursue the second route.

The structure of the paper is as follows. After reviewing the HLL method, we find a modified MHD system which allows $\nabla \cdot \mathbf{B} \neq 0$. We then apply the HLL method to this modified MHD system and show its positivity by numerical experiments. After having found this positive and conservative method, we show that to increase its accuracy, it can be hybridized by any other method of computing the interface fluxes while retaining positivity. As an example we hybridize it with Roe's method. The resulting method is identical with Roe's method whenever Roe's method is in no danger of violating positivity, but reverts to HLL in the remaing cases. This method, therefore, has the well-known accuracy properties of Roe's method, but has cured its positivity problem.

2 The HLL method

The HLL scheme without a source term has been given elsewhere [3, 4]. We present the derivation here, also including a source term.

Consider a one-dimensional hyperbolic system [11]

$$\frac{\partial u}{\partial t} = -\frac{\partial f}{\partial x} + s \tag{2}$$

where $u = u(x, t)$ is the solution vector, $f = f(u)$ is the flux vector and $s = s(x, t)$

is the source term. The initial condition is of the Riemann problem type,

$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases} . \tag{3}$$

Let b_L and b_R be the minimum and maximum wavespeed, respectively, so that $u(x,t) = u_L$ for $x < b_L t$ and $u(x,t) = u_R$ for $x > b_R t$. Let τ be the timestep and $L = (b_R - b_L)\tau$ the length of the interval on which $u(x,\tau)$ can differ from u_L and u_R. We first assume that $b_L \leq 0$ and $b_R \geq 0$ and return to the other cases below. We denote the spatial average over interval L by $\bar{u}(t)$,

$$\bar{u}(t) \equiv \frac{1}{L} \int_{b_L\tau}^{b_R\tau} dx u(x,t). \tag{4}$$

Integrating (2) from 0 to τ and taking the spatial average over L yields

$$\bar{u}(\tau) - \bar{u}(0) = -\frac{\tau}{L}\left(f_R - f_L\right) + \int_0^\tau dt \bar{s}(t) \tag{5}$$

where $f_R = f(u_R)$ and $f_L = f(u_L)$.

The spatial average at the initial moment $\bar{u}(0)$ is easy to compute by using (3) and the assumptions $b_L \leq 0$, $b_R \geq 0$, we obtain

$$\bar{u}(0) = \frac{b_R u_R - b_L u_L}{b_R - b_L}. \tag{6}$$

Thus we can write $\bar{u}(\tau) = u_m + \Delta u_m$, where

$$u_m = \frac{b_R u_R - b_L u_L - (f_R - f_L)}{b_R - b_L} \tag{7}$$

and

$$\Delta u_m = \int_0^\tau dt \bar{s}(t). \tag{8}$$

The state $\bar{u}(\tau)$ is the average of the exact solution of the Riemann problem. If the set of physical states of the hyperbolic system (states with positive density and pressure) is convex, any average of physical states is a physical state [4]. Both Euler and MHD equations are convex in this sense. Convexity means that if u_1 and u_2 are physical states, then $u = (1-\lambda)u_1 + \lambda u_2$ is also, for $0 \leq \lambda \leq 1$. To show the convexity of MHD, we can write the density and pressure of the average state u as

$$\begin{aligned} \rho &= (1-\lambda)\rho_1 + \lambda\rho_2 \\ P &= (1-\lambda)P_1 + \lambda P_2 + (\gamma-1)\lambda(1-\lambda)\left[\frac{1}{2}\frac{\rho_1\rho_2}{\rho}(\Delta\mathbf{v})^2 + \frac{(\Delta\mathbf{B})^2}{2\mu_0}\right] \end{aligned} \tag{9}$$

from which it is easily seen that $\rho > 0$ and $P > 0$ if $\rho_1 > 0, \rho_2 > 0, P_1 > 0, P_2 > 0$ and $0 \leq \lambda \leq 1$. Here, $\Delta\mathbf{v}$ and $\Delta\mathbf{B}$ are the jumps between states u_1 and u_2 in the velocity and magnetic field.

For convex hyperbolic systems we get the important results that if the exact solutions of any Riemann problem is physical, the the HLL middle state $\bar{u}(\tau)$ is also physical. Likewise, if the HLL middle state $\bar{u}(\tau)$ turns out to be nonphysical, we can infer that the exact Riemann problem solution must also be nonphysical, or else the bounds of the wavespeeds b_L and b_R have been underestimated.

Next we need the expression for the HLL interface flux F_{HLL} [4]. The flux F_{HLL} does not depend on Δu_m but only on u_m, u_L and u_R. One way of computing F_{HLL} is to consider the cell L which resides left of the interface. Since F_{HLL} can not depend on the data on cells which are left from cell L, we can assume that these have the same state as cell L, i.e. that there is no jump on the left hand interface of cell L. Because the numerical flux function F_{HLL} must equal the analytic flux $f(u)$ if there is no jump [11], the flux entering cell L from the left must be equal to f_L. After time τ, the solution in cell L is equal to u_L on the left side of the cell and to u_m on the right side of the cell. The length of the interval where the solution is u_m is $-b_L\tau$. Thus we can write for the cell average after time τ

$$\bar{u}_L(\tau) = \frac{(\Delta x_L + b_L\tau)u_L - b_L\tau u_m}{\Delta x_L} = u_L - \frac{\tau}{\Delta x_L}(F_{\mathrm{HLL}} - f_L) \tag{10}$$

from which we obtain

$$F_{\mathrm{HLL}} = f_L + b_L(u_m - u_L). \tag{11}$$

Considering the right hand cell R in a similar way we obtain an alternative expression for F_{HLL},

$$F_{\mathrm{HLL}} = f_R + b_R(u_m - u_R). \tag{12}$$

The equivalence of Eqs. (11) and (12) can be shown easily by using (7). By substituting u_m from (7) we can also write

$$F_{\mathrm{HLL}} = \frac{b_L b_R(u_R - u_L) + b_R f_L - b_L f_R}{b_R - b_L}. \tag{13}$$

The above formulas for the HLL flux hold for $b_L \leq 0 \leq b_R$. If $b_L \leq b_R < 0$, we have simply $F_{\mathrm{HLL}} = f_R$. Likewise, if $0 < b_L \leq b_R$ we have $F_{\mathrm{HLL}} = f_L$.

The contribution of the source term Δu_m to cells L and R must be computed by taking into account how far the waves which are limited by $b_L\tau$ and $b_R\tau$ have propagated within each cell. On cell L the average $\bar{u}_L(\tau)$ must therefore be incremented by $\Delta\bar{u}_L(\tau)$,

$$\Delta\bar{u}_L(\tau) = \frac{\Delta x_{mL}}{\Delta x_L}\Delta u_m \tag{14}$$

where $\Delta x_{mL} = (\min(0, b_R) - \min(0, b_L))\tau$. On cell R the corresponding modification is

$$\Delta\bar{u}_R(\tau) = \frac{\Delta x_{mR}}{\Delta x_R}\Delta u_m \tag{15}$$

where $\Delta x_{mR} = (\max(0, b_R) - \max(0, b_L))\tau$. These formulas are valid for the cases $b_L > 0$ and $b_R < 0$ also. For each cell, one has to use both Eq. (14) and (15).

3 Magnetic monopoles in MHD

Magnetic monopoles were introduced in MHD in the pioneering work of Powell [15, 16]. Similar equations had appeared already much earlier, but in a different context [5]. Powell's interest was not so much in deriving a positive method but rather to keep $\nabla \cdot \mathbf{B}$ small without using elliptic cleaning. He reports that his equations keep $\nabla \cdot \mathbf{B} \approx 0$ within truncation error and no elliptic cleaning is necessary [16].

Here we shall consider magnetic monopoles in MHD by going back to fundamentals first. A proper generalization of Maxwell's equations when magnetic monopoles are present is given by

$$\nabla \cdot \mathbf{E} = \rho_e/\varepsilon_0$$

$$
\begin{aligned}
-\nabla \times \mathbf{E} &= \mathbf{j}_m + \frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= \rho_m \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j}_e + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t},
\end{aligned} \tag{16}
$$

where the subscript e refers to electric charges and currents and m refers to magnetic charges and currents [7]. These equations remain invariant in a global duality transform which mixes electric and magnetic fields [7]:

$$
\begin{aligned}
&\mathbf{E} \to \mathbf{E}' = \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha \\
&\mathbf{B} \to \mathbf{B}' = -\frac{1}{c}\mathbf{E}\sin\alpha + \mathbf{B}\cos\alpha
\end{aligned} \tag{17}
$$

provided that the charges transform as

$$
\begin{aligned}
q_e \to q_e' &= q_e \cos\alpha + \frac{1}{\mu_0 c} q_m \sin\alpha \\
q_m \to q_m' &= -\frac{1}{\varepsilon_0 c} q_e \sin\alpha + q_m \cos\alpha.
\end{aligned} \tag{18}
$$

The charge densities ρ_e, ρ_m and current densities $\mathbf{j}_e$, $\mathbf{j}_m$ transform in the same way as q_e, q_m. In these formulas, α is an arbitrary parameter, which does not depend on position and time. If we require that the Lorentz force, which has the expression

$$
\mathbf{F} = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{19}
$$

for a purely electric charge q_e, is invariant under a duality transformation, we see by making a duality transform with $\alpha = \pi/2$ that the Lorentz force acting on a purely magnetic charge q_m is

$$
\mathbf{F} = \frac{1}{\mu_0} q_m (\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}). \tag{20}
$$

Thus the generalization of the Lorentz force density is

$$
\mathbf{f} = \rho_e \mathbf{E} + \mathbf{j}_e \times \mathbf{B} + \frac{1}{\mu_0}\rho_m \mathbf{B} - \varepsilon_0 \mathbf{j}_m \times \mathbf{E}. \tag{21}
$$

Even without resorting to the duality invariance, it is intuitively obvious that a magnetic field must exert a force on a magnetic charge, in a similar vein as an electric field exerts an electrostatic force on an electric charge.

The $\mathbf{j}_m \times \mathbf{E}$ term is of the order $(v/c)^2$ times smaller than the term proportional to $\mathbf{B}$ and thus can be dropped in the nonrelativistic case, which the MHD equations represent anyway (the displacement current term proportional to $\partial \mathbf{E}/\partial t$ is dropped from Ampere's law). In MHD, modifications arise in Faraday's law, where the magnetic current term $\mathbf{j}_m$ must be added, and in the momentum equation, where the "magnetostatic" force density $\rho_m \mathbf{B}/\mu_0$ appears. Thus, the MHD system in the primitive variable formulation in the presence of magnetic monopoles is written as

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \mathbf{v}) \\
\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} &= \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla P + \frac{1}{\mu_0}\mathbf{B}\nabla\cdot\mathbf{B} \\
\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\left(P\rho^{-\gamma}\right) &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \mathbf{j}_m.
\end{aligned} \tag{22}
$$

To close the system of equations, the magnetic current $\mathbf{j}_m$ must be expressed in terms of the other variables. The "minimal" choice is

$$\mathbf{j}_m = \rho_m \mathbf{v} = (\nabla \cdot \mathbf{B})\mathbf{v}, \tag{23}$$

where it has been assumed that all magnetic charges move with the same velocity $\mathbf{v}$ as the plasma flow and there is no difference between positive and negative magnetic charges. We will use this expression in what follows. Since the only purpose of including magnetic monopoles is to obtain a positive system of equations, it does not really matter what kind of particles the magnetic charges consist of. Any physically consistent assumption will do, since from general physical grounds we expect that that for any system of particles, including electric and magnetic charges, the corresponding fluid equations ought to have a positive pressure solution. For example, we could assume that the magnetic charges are bound to ions so that there are positive ions with positive magnetic charge and positive ions with negative magnetic charge and negatice electrons with zero magnetic charge.

Substituting equation (23) in the MHD equations above and writing them in conservative form yields, after some vector algebra (remembering that $\nabla \cdot \mathbf{B} \neq 0$):

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{p} &= 0 \\
\frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot \left[\frac{\mathbf{pp}}{\rho} + \left(P + \frac{B^2}{2\mu_0} \right) \mathbf{I} - \frac{1}{\mu_0} \mathbf{BB} \right] &= 0 \\
\frac{\partial U}{\partial t} + \nabla \cdot \left[\left(U + P + \frac{B^2}{2\mu_0} \right) \mathbf{v} - \frac{1}{\mu_0} (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \right] &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{vB} - \mathbf{Bv}) &= -\mathbf{v} \nabla \cdot \mathbf{B}
\end{aligned} \tag{24}$$

where $\mathbf{I}$ is the unit dyad.

Eqs. (24) are identical to those derived by Powell [15, 16], except that in Powell's version, there are additional source terms proportional to $\nabla \cdot \mathbf{B}$ also in the momentum and energy equation. The difference between our equations and Powell's is due to the fact that we include the "magnetostatic" force density $\mathbf{B} \nabla \cdot \mathbf{B}/\mu_0$ in the Lorentz force expression. If one leaves out this term from the momentum equation, one obtains Powell's version.

Our equations conserve momentum and energy, whereas in Powell's equations, momentum and energy are not conserved if $\nabla \cdot \mathbf{B} \neq 0$. We think that the presence of magnetic monopoles should not break the conservation of total energy and momentum, because energy and momentum should be conserved in any physical system. Another way to see this is that if the magnetostatic force is left out, there is no force at all acting on magnetic charges and they move as "godlike" particles, which is unphysical. Thus we believe that our version is the "correct" one, although correctness can not be subjected to experimental testing in this case because magnetic monopoles have not been found in nature.

In both Eqs. (24) and Powell's, the source terms in Faraday's law are similar. Thus we can make the same conclusion as Powell [15, 16] did: $\nabla \cdot \mathbf{B}$ is convected as a passive scalar,

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) + \nabla \cdot (\mathbf{v} \nabla \cdot \mathbf{B}) = 0. \tag{25}$$

Since these equations conserve mass, momentum and energy, we call them conservative. The fact that the volume integrals of B_x, B_y and B_z are not conserved if $\nabla \cdot \mathbf{B} \neq 0$ is not physically important, as was also mentioned by Linde [12].

The linearized eigenstructure of Eqns. (24) is quite similar to Powell's equations [16]. The eigenvalues are the same, and the "divergence wave" exists. The divergence wave eigenvectors differ, the other seven eigenvectors are the same.

4 HLL method for monopole-MHD

We must now find a proper discretization for the source term $\mathbf{s} = -\mathbf{v}\nabla \cdot \mathbf{B}$ in Faraday's law. In a one-dimensional Riemann problem $\nabla \cdot \mathbf{B} = \partial B_x/\partial x$.

The magnetic part of the HLL source term from Eq. (8) is (the other parts are zero)

$$\begin{aligned}
\Delta \mathbf{B}_m &\equiv \int_0^\tau dt \bar{\mathbf{s}}(t) \approx \int_0^\tau dt \frac{1}{L} \int_{b_L\tau}^{b_R\tau} dx \left(-\mathbf{v}_m \frac{\partial B_x}{\partial x}\right) \\
&= \int_0^\tau dt \frac{1}{L} \int_{b_L t}^{b_R t} dx \left(-\mathbf{v}_m \frac{\partial B_x}{\partial x}\right) \\
&= \frac{1}{b_R - b_L} \int_0^\tau \frac{dt}{\tau} \left(-\mathbf{v}_m\right) \Delta B_x \\
&= -\frac{\mathbf{v}_m \Delta B_x}{b_R - b_L}. \qquad (26)
\end{aligned}$$

We approximated $\mathbf{v}$ in the range $[b_L t, b_R t]$ by $\mathbf{v}_m$, where $\mathbf{v}_m$ is the velocity computed from the HLL middle state u_m, Eq. (7). The range of x integration could be reduced from $[b_L\tau, b_R\tau]$ to $[b_L t, b_R t]$, because $\partial B_x/\partial x = 0$ for $x < b_L t$ and $x > b_R t$ for each t satisfying $0 < t \leq \tau$. We also needed to assume that B_x is continuous for $0 < t \leq \tau$ (for $t = 0$, B_x is of course discontinuous). We do not need to know exactly how B_x varies in the interval $[b_L t, b_R t]$.

For the HLL method to be positive, we must find expressions for the lower and upper bounds b_L and b_R for the wavespeeds of the exact Riemann problem solution. We are not aware of rigorous bounds for MHD, instead we do the following:

$$\begin{aligned}
\hat{a}^2 &= \frac{\gamma \max(P_L, P_R)}{\min(\rho_L, \rho_R)} \\
\hat{v}_A^2 &= \frac{\max(B_L^2, B_R^2)}{\mu_0 \min(\rho_L, \rho_R)} \\
\hat{v}_{Ax}^2 &= \frac{\max(B_{Lx}^2, B_{Rx}^2)}{\mu_0 \min(\rho_L, \rho_R)} \\
\hat{v}_f^2 &= \frac{1}{2}\left[\hat{a}^2 + \hat{v}_A^2 + \sqrt{\left(\hat{a}^2 - \hat{v}_A^2\right)^2 + 4\hat{a}^2\left(\hat{v}_A^2 - \hat{v}_{Ax}^2\right)}\right]. \qquad (27)
\end{aligned}$$

after which the wavespeed bounds are computed from

$$\begin{aligned}
b_L &= \min(v_{xL}, v_{xR}) - \hat{v}_f \\
b_R &= \max(v_{xL}, v_{xR}) + \hat{v}_f. \qquad (28)
\end{aligned}$$

These wavespeed bounds guarantee positivity according to our numerical verification (see below). They probably overestimate the true wavespeeds somewhat, which may increase the diffusion of our HLL method [4]. However, in most cases the jump between the left and right states is not large and in these cases b_L and b_R approach to the limits of the state eigenvalues, and the non-sharpness of the bounds is numerically insignificant. Below we will show how to hybridize the method with the Roe method, which makes a modest increase of HLL diffusion quite tolerable because the HLL method is reverted to in only rare cases then.

We have performed extensive numerical searches and experimentation to find counterexamples to the positivity of this method. No counterexamples have been found, but the method has produced a positive solution in every numerical test that we have thus far carried out. Among the tests we have performed are the following:

- Computing $u_m + \Delta u_m$ for Riemann problems with random positive left and right states. The total number of random problems scanned in this way was over ten million.
- The test problems of Einfeldt *et al.*[4] were used. We also tried to increase the magnitude of the velocity components in the shear and rarefaction wave examples, and tried the same examples with a large jump in B_x.
- The magnetic shock tube problem of Brio and Wu [1] was solved, and the same problem with a large jump in B_x.
- Problems with randomly generated initial conditions were solved. The initial data contained random jumps in B_x also.

All of the above tests readily report negative pressures if the Faraday law source term is dropped, unless the B_x jumps are removed as well. We believe that these tests are extensive enough to show beyond reasonable doubt that the method is positive under all circumstances. We also discretized Powell's source terms in the same way and found that the method is not always positive. For example, the Riemann problem

$$\begin{aligned} \text{leftstate} &= [\rho = 1, v_y = 10, P = 0.1, B_x = -1] \\ \text{rightstate} &= [\rho = 1, v_y = 10, P = 0.1, B_x = 1] \end{aligned} \tag{29}$$

($\gamma = 5/3, \mu_0 = 1$, the unlisted components are zero) breaks Powell's method. In principle it is of course possible that some other discretization would render Powell's method positive.

5 Hybridization with Roe's method

Assume that F is a numerical flux function computed by any method, and we want to find if it gives guaranteed positive updated left and right states. Assume that the wavespeed bounds again satisfy $b_L \le 0 \le b_R$ (if not, then hybridization is unnecessary since the flux is then given by f_L or f_R). We take a "virtual" cell of length $-b_L\tau$ on the left side of the interface and another virtual cell of length $b_R\tau$ on the right side. The virtual cells are smaller than the real cells because the Courant number is smaller than one. From the conservation law (2) one easily obtains the following expressions for the updated virtual cell states:

$$\begin{aligned} u_L^{\text{virt}} &= u_L + \frac{F - f_L}{b_L} + \Delta u_m \\ u_R^{\text{virt}} &= u_R + \frac{F - f_R}{b_R} + \Delta u_m. \end{aligned} \tag{30}$$

To hybridize flux F with the HLL flux F_{HLL} (13) one only has to check whether the states u_L^{virt} and u_R^{virt} are physical. If they are, one can safely use the flux F. If either of them is not physical, there is a danger of using F, and one has to replace the interface flux by F_{HLL} to ensure positivity.

We have carried out the hybridization for the Roe-type interface flux [1, 18, 15, 16, 12] and run the same test problems as were used to show the positivity of the HLL method in the previous section. The hybridized method was positive in all cases and it gave identical results with the Roe method in cases where the Roe method was in no danger of producing nonphysical states. In this test we used the linearization given by Powell [16].

In principle one could use the true cell widths rather than the virtual cells when checking positivity. By doing this, the flux F would be abandoned only when it would with certainty produce a nonphysical state. We think that it may be good for roundoff error and for other reasons do revert to the HLL method a little bit earlier, because when the virtual states become nonphysical, the linearized Roe method has already been pushed to a parameter range where its accuracy has been lost, so one could equally well use the HLL method in these cases. Our experience shows that in typical physical problems, reverting to HLL occurs only rarely and thus the vast majority of Riemann problems solved are solved by the Roe method.

6 Conclusions and future work

We have found a positive and conservative method for ideal MHD equations by first generalizing the MHD equations to allow for magnetic monopoles. Since positive and conservative methods are also numerically stable, this is an important advance in MHD simulation.

The usual MHD equations are based on Maxwell's equations which do not allow magnetic monopoles. If one tries to use an initial condition which has $\nabla \cdot \mathbf{B} \neq 0$ with the ordinary MHD equations, the equations punish us by sometimes giving a negative pressure.

We derived monopole-MHD equations by starting from generalized Maxwell's equations which are invariant under duality transforms that mix electric and magnetic fields and charges. The duality transform implies a generalized expression for the Lorentz force which contains the magnetostatic force acting on magnetic charges. The monopole-MHD equations are the same as the usual MHD equations, except that there is the source term $-\mathbf{v}\nabla \cdot \mathbf{B}$ on the right-hand side of Faraday's law.

We showed numerically that the monopole-MHD equations are HLL-positive for any physical initial data if one discretizes the source term in a specific way. By this we mean that the HLL middle state $u_m + \Delta u_m$, where Δu_m is the Faraday source term contribution, is positive. We did not try to investigate the positivity of the exact solutions of monopole-MHD Riemann problems, but we are inclined to conjecture that there is a positive pressure solution (or vacuum) for these Riemann problems for any physical left and right states.

It is interesting to note that a seemingly esoteric subject such as magnetic monopoles appears to play a key role in developing a robust, positive and conservative numerical method for MHD.

The derived positive and conservative method is fully robust, but rather diffusive. To improve on this, we noted that it is possible to hybridize the developed HLL-monopole-MHD method with any method that produces interface fluxes. One just has to check the positivity of the left and right "virtual" states, and replace the interface flux with the HLL flux if either of them is nonphysical. This operation is fast to perform and can be used to introduce positivity to any non-positive numerical flux function. As an example, we carried out the hybridization for the first-order Roe method. The hybridization of second or higher order Roe methods could be similarly studied.

The hybridization method presented in this paper is not limited to MHD. It works for any hyperbolic system of conservation laws and a source term, for which a positive (HLL-type) Riemann solver is known. On the other hand, in the absence of a source term, the HLL Riemann solver is positive, if the exact solution of any Riemann problem is positive and if the set of physical states is convex. Thus besides MHD, the hybridization could be applied e.g. to any conservative system whose Riemann problems are well-posed and whose set of physical states is convex.

We have not yet considered in detail what happens if the background magnetic field is analytically separated in the manner first done by Tanaka [20, 21]. In principle, one doesn't have to separate the background field if our method is really positive under all circumstances. However, it might still be wise to do so to avoid roundoff error problems. Separating the background field might also be a more accurate way, even if there is no difference in positivity.

In the near future we will implement our scheme in our three-dimensional global MHD simulation. Going from one to three dimensions does not affect positivity since the 3D solution is computed by breaking the problem down in one-dimensional Riemann problems. We will then also see how small $\nabla \cdot \mathbf{B}$ remains and whether elliptic cleaning is needed or not.

References

[1] M. Brio, M. and C.C. Wu, An upwind differencing scheme for the equations of ideal magnetohydrodynamics, *J. Comp. Phys., 75,* p. 400, 1988.

[2] J. Croisille, R. Khanfir and G. Chanteur, Numerical simulation of the MHD equations by a kinetic-type method, *J. Sci. Comput.,* **10**, p. 81, 1995.

[3] B. Einfeldt, On Godunov-type methods for gas dynamics, *SIAM J. Numer. Anal.,* **25**, p. 294, 1988.

[4] B. Einfeldt, C.D. Munz, P.L. Roe and B Sjögreen, On Godunov-type methods near low densities, *J. Comp. Phys., 92,* p. 273, 1991.

[5] S.K. Godunov, On symmetric form of the equations of magnetohydrodynamics, Technical report, USSR Academy of Sciences Computer Center, 1972. English translation available at `http://www.engin.umich.edu/research/cfd/publications/papers/Godunov.ps.gz`.

[6] A. Harten, P.D. Lax and B. van Leer, On upstream differencing and Godunov-type schemes for hyperbolic conservation laws, *SIAM Rev.,* **25**, p. 35, 1983.

[7] J.D. Jackson, Classical electrodynamics, Second edition, Wiley, New York, 1975.

[8] P. Janhunen, GUMICS-3 – a global ionosphere-magnetosphere coupling simulation with high ionospheric resolution, Proc. Environmental Modelling for Space-Based Applications, 18-20 Sept. 1996, ESTEC, The Netherlands, ESA SP-392, 1996.

[9] P. Janhunen, On the possibility of using an electromagnetic ionosphere in global MHD simulations, *Ann. Geophys., 16,* p. 397, 1998.

[10] P. Janhunen, O. Amm and R. Greenwald, Adaptive mesh in global magnetosphere simulation: Comparisons with global convection, 1998 Lisbon, Portugal Cambridge Symposium Workshop Proceedings, MIT, 1998.

[11] R.J. LeVeque, Numerical methods for conservation laws, Birkhäuser, 1992.

[12] T.J. Linde, A three-dimensional adaptive multifluid MHD model of the heliosphere, *Ph D thesis,* Univ. Michigan, Aerospace Engineering, Ann Arbor, Michigan, 1998.

[13] Moschetta, J.-M. and D.I. Pullin, A robust low diffusive kinetic scheme for the Navier-Stokes/Euler equations, *J. Comp. Phys., 133,* 193–204, 1997.

[14] R.S. Myong, Theoretical and computational investigations of nonlinear waves in magnetohydrodynamics, *Ph D thesis,* Univ. Michigan, Aerospeca Engineering, Ann Arbor, Michigan, 1996.

[15] K.G. Powell, An approximate Riemann solver for magnetohydrodynamics (that works in more than one dimension), *ICASE Report* 94–24, 1994.

[16] K.G. Powell, Solution of the Euler and magnetohydrodynamic equations on solution-adaptive Cartesian grids, in *Computational Fluid Dynamics,* Lecture Series 1996-06, VKI, March, 1996.

[17] J. Raeder, R.J. Walker and M. Ashour-Abdalla, The structure of the distant geomagnetic tail during long periods of northward IMF, *Geophys. Res. Lett., 22,* p. 349, 1995.

[18] P.L. Roe and D.S. Balsara, Notes on the eigensystem of magnetohydrodynamics, *SIAM J. Appl. Math.,* **56**, 1, p. 57, 1996.

[19] M. Schmidt-Voigt, Time-dependent MHD simulations for cometary plasmas, *Astron. Astrophys,* **210**, p. 433, 1989.

[20] T. Tanaka, Finite volume TVD scheme on an unstructured grid system for three-dimensional MHD simulation of inhomogeneous systems including strong background potential fields, *J. Comp. Phys., 111,* p. 381, 1994.

[21] T. Tanaka, Generation mechanisms for magnetosphere-ionosphere current systems deduced from a three-dimensional MHD simulation of the solar wind-magnetosphere-ionosphere coupling processes, *J. Geophys. Res., 100,* p. 12057, 1995.