Magnetically dominated plasma models of ball lightning

Pekka Janhunen
Finnish Meteorological Institute, Geophysics Department
21.12. 1989

Recently, ball lightning models based on MHD force balance equation have been proposed. An upper bound for the magnetic energy for these models is presented. The possibility of weakly ionized plasma models is considered with estimates on lifetime and energy concent. The lifetime is found to be too short, if the electron-neutral and ion-neutral effective collision frequencies behaver in the usual way. The possibilities to get around these restrictions are briefly analyzed.

I. INTRODUCTION

In trying to construct a model for ball lightning one notes that while it is possible to explain each individual property of ball lightning alone, it has been thus far too difficult to have a single theory that would reproduce all relevant properties simultaneously. Perhaps the most important things to explain are the energy storage mechanism and stability. In this paper, models with magnetic energy storage (i.e. the energy of the ball is mainly the magnetic field energy) are considered.

From the usual size of ball lightning and reported total energies [1] one can obtain an order-of-magnitude estimate for the magnetic field. Doing this, one can find that fields of one tesla or even more must occur inside ball lightning, if the assumption of magnetic energy storage is made.

Recently, stationary axially symmetric solutions of the magnetohydrodynamic force-balance equation $\nabla p = \mathbf{j} \times \mathbf{B}$ has been found[2]. The force balance equation can be reduced to a partial differential equation of two variables containing two arbitrary functions of single real argument. The analytical solutions presented correspond to a special choice of these arbitrary functions in order to make the equation linear. However, there exists an upper bound for the magnetic energy for all fields satisfying the MHD equations, as we now show.

II. THE MHD FORCE BALANCE EQUATION

Consider a stationary MHD plasma sphere imbedded in atmospheric pressure p_0. It is assumed that the exterior domain is non-ionized air. The gravity is neglected because relevant magnetic fields are strong enough to yield a Lorentz force dominating the gravitational force. The relevant equations are the force balance equation and Maxwell's equations for the magnetic field[2]:

$$\nabla p = \mathbf{j}\times\mathbf{B}, \quad \nabla\times\mathbf{B} = \mu_O\mathbf{j}, \quad \nabla\cdot\mathbf{B} = 0. \tag{1}$$

Boundary conditions are:

$$\mathbf{B} = 0, \quad p = p_O,\ \partial p/\partial n = 0 \qquad \text{at the boundary} \tag{2}$$

The boundary is assumed to be a spherical surface of (finite) radius r_O. The parameter r_O determines the length scale for the problem.

All solutions of Eq. (1) have the property that the total magnetic field energy has an upper limit, if the solution is confined to a finite region of space. From the vector identity

$$\mathbf{r}\cdot(\nabla\times\mathbf{B})\times\mathbf{B} = \nabla\cdot(\mathbf{B}(\mathbf{r}\cdot\mathbf{B})) + (1/2)\ B^2 - (1/2)\nabla\cdot(\mathbf{r}B^2) \tag{3}$$

we obtain the magnetic energy content of any volume V as

$$W_m = \int_V d\mathbf{r}\,\frac{1}{2\mu_o}B^2 = \int_V d\mathbf{r}\ \mathbf{r}\cdot\mathbf{j}\times\mathbf{B} + \frac{1}{\mu_o}\oint_{\partial V} d\mathbf{S}\cdot\left[\frac{1}{2}\mathbf{r}B^2 - \mathbf{B}(\mathbf{r}\cdot\mathbf{B})\right]. \tag{4}$$

The surface term vanishes because of boundary conditions (2). Then, by using Eq. (1), we have

$$W_m = \int_V d\mathbf{r}\ \mathbf{r}\cdot\nabla p = \oint_{\partial V} d\mathbf{S}\cdot\mathbf{r}p - 3\langle p\rangle V = 3(p_o - \langle p\rangle)\,V \tag{5}$$

where we have <p> stands for the average pressure inside V. Again, we have used the boundary conditions (2). Because the pressure is always non-negative, we get the value $3p_OV$ as an upper bound for ball lightning magnetic energy, where V is the volume of the plasma sphere. This upper bound is of the same order as the energy released in an implosion of a vacuum tube of volume V. Typical values for V range from 7.2 to 17 10^{-3} m^3 [3], which yields 2..5 kJ for the upper bound of energy. On the other hand, typical values for the observed energies are from 12 to 30 kJ [3]. In extreme cases, the energies have been reported to be as high as 1000 kJ [4].

We are now in the position to decide whether the MHD models can describe real ball lightning or not (as well as total energy is concerned). The upper bound for MHD models is about five times smaller than average observed value. At this stage, let us note that the real energy content for MHD models is probably clearly less than the upper bound, because, apart from being non-negative, the pressure inside ball lightning is probably well above zero the system to be stable at all. For the explicit solution, the average pressure inside MHD ball lightning has been found to be only slightly (about 10 percent) smaller than the atmospheric value [2]. If we adobt this as a general rule, the MHD energy will be about 50 times smaller than the averabe observed ones.

From this we conclude that if we use the MHD model for high energy forms of ball lightning, the principal energy storage is not the magnetic field. The MHD equations give no hint of what that storage mechanism could be.

III. THE POSSIBILITY OF "LOWER TEMPERATURE" MODELS

In MHD models, it is necessary to assume that the ionization degree inside plasma sphere is high. At atmospheric pressures, this will yield very high ($10^4..10^5$ K) temperatures. Temperatures that high will probably lead to rapid energy losses due to convection and radiation, giving rise to more rapid decay than the characteristic time $\tau = \mu_0 \sigma r_0^2$, where σ is the conductivity of the plasma. Furthermore, the sensation of heat is seldom present in ball lightning events [1].

If, on the other hand, the plasma sphere is only weakly ionized, the MHD description is no longer valid, because collisions of plasma particles with neutrals has to be taken into account. A weakly ionized plasma is described in the stationary state by the generalized Ohm's law

$$\mathbf{j} = \sigma_{||}\mathbf{E}'_{||} + \sigma_P \mathbf{E}'_{\perp} + \sigma_H \mathbf{B}\times\mathbf{E}'/B \qquad (6)$$

where $\sigma_{||}$, σ_P and σ_H are called the direct, Pedersen and Hall conductivity, respectively. $\mathbf{E}'$ is the effective electric field, given in the stationary state by the expression

$$\mathbf{E}' = -\nabla\Phi + \mathbf{v}\times\mathbf{B} \qquad (7)$$

where Φ is the electric potential and $\mathbf{v}$ is the average velocity of neutral gas. The direct resistivity is entirely due to collisions, and has the same value as the scalar conductivity if no magnetic field is present. For simplicity, we assume $\sigma_{||} = \infty$. The Pedersen and Hall conductivities are given by the expressions

$$\sigma_P = \frac{en}{B}\left(\frac{\nu_i\Omega_i}{\nu_i^2 + \Omega_i^2} - \frac{\nu_e\Omega_e}{\nu_e^2 + \Omega_e^2}\right) \qquad (8)$$

$$\sigma_H = \frac{en}{B}\left(-\frac{\Omega_i^2}{\nu_i^2 + \Omega_i^2} + \frac{\Omega_e^2}{\nu_e^2 + \Omega_e^2}\right) \qquad (9)$$

where n is the electron (and ion) density and $\nu_{i,e}$ and $\Omega_{i,e}$ are the effectice collision frequencies with neutrals and Larmor frequencies for ions and electrons, respectivly.

Let us assume that ν_i and ν_e are of the same order, at least when compared with the large ratio Ω_e/Ω_i, which is of the same order as the ion-electron mass ratio. Then it is possible to consider three limiting cases: 1) large B when $\nu_{i,e} \ll \Omega_i \ll |\Omega_e|$, 2) "intermediate" B when $\Omega_i \ll \nu_{i,e} \ll |\Omega_e|$, and 3) small B when $\Omega_i \ll |\Omega_e| \ll \nu_{i,e}$. Analysis of the relative importances of the Pedersen and Hall conductivities then reveals that the Pedersen conductivity dominates the Hall conductivity in all cases except 2), when the Hall conductivity dominates. More generally, if ν_i is not much smaller than ν_e, we obtain for the relative importance of the two conductivities the expression

$$\frac{\sigma_H}{\sigma_P} = \frac{1}{\frac{\Omega_i}{\nu_i} + \frac{\nu_e}{|\Omega_e|}} \qquad (10)$$

Let us consider the Hall dominated case in some more detail. In this case the motion of the electrons is determined by the magnetic field and they obey the **E**x**B** drift. Ions, on the other hand, are not magnetized, i.e. their motion is collision dominated. Because of this, the electric current is carried mainly by electrons, which means that

$$\mathbf{j} = -en\mathbf{v}_e = \frac{en}{B^2}\,\mathbf{B}\times\nabla\Phi \qquad (11)$$

in the stationary state. In addition, a magnetic field aligned current $\mathbf{j}_{||}$ may flow. From IV Maxwell's equation it then follows, when $\sigma_{||} = \infty$, that

$$\mathbf{B}\times\nabla\times\mathbf{B} = \mu_o en\ \nabla\Phi. \qquad (12)$$

On the other hand, it is easy to see that if **B** satisfies (12), then there exists such $\mathbf{j}_{||}$ that Maxwell's equations are satisfied (and $\mathbf{B}\times\mathbf{j}_{||} = 0$).

IV. THE COLD PLASMA BALL LIGHTNING

To get rid of the upper bound associated with the MHD equations, we suggest the "cold" plasma model. In the stationary state, this is described by a set of equations analogous to Eq (1), but the term ∇p replaced by $(1/c^2)\ (\nabla^2\Phi)\ \nabla\Phi$, where Φ is the electric scalar potential. Redefining Φ so that c=1, we have the system of equations

$$\mathbf{B}\times(\nabla\times\mathbf{B}) = (\nabla^2\Phi)\nabla\Phi, \qquad \nabla\cdot\mathbf{B} = 0 \qquad (5)$$

We have assumed that the pressure term is negligible compared with the potential term. This equation is rather similar in structure to the corresponding MHD equation (1). Let us assume that the plasma ball is axially symmetric and let us use spherical coordinates (r,θ,φ). Then, we can write **B** in the following form, which is automatically divergence-free:

$$\mathbf{B}(r,\theta) = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} A(r,\theta)\, \widehat{\mathbf{e}_r} - \frac{1}{r \sin\theta} \frac{\partial}{\partial r} A(r,\theta)\, \widehat{\mathbf{e}_\theta} + \frac{1}{r\sin\theta} b_\varphi(r,\theta)\, \widehat{\mathbf{e}_\varphi} \tag{6}$$

Then the φ-component of the force balance equation implies that $\Phi(r,\theta) = \Phi(A(r,\theta))$ and $b_\varphi(r,\theta) = b_\varphi(A(r,\theta))$. So, there are two arbitrary functions of one real argument also in the cold plasma case. When these relations are substituted to the two remaining components of the force balance equation, it is found that the resulting equations are in fact equivalent. Thus, we obtain a single equation for the flux function A(r,θ):

$$\begin{aligned} & r^2 \sin\theta\, (1 - r^2 \sin^2\theta\, \gamma(A)) \left[\partial_r^2 A + \frac{1}{r^2} \partial_\theta^2 A\right] \\ & \quad - 2r^3 \sin^3\theta\, \gamma(A)\, \partial_r A - \cos\theta \left[\, 1 + r^2 \sin^2\theta\, \gamma(A)\right] \partial_\theta A \\ & \quad - \frac{1}{2} r^4 \sin^3\theta\, \gamma'(A) \left[(\partial_r A)^2 + \frac{1}{r^2} (\partial_\theta A)^2\right] + r^2 \sin\theta\, \beta'(A) = 0 \end{aligned} \tag{7}$$

where the dot denotes differentiation with respect to the argument, A = A(r,θ), and we have defined $\beta(A) = (1/2)\, b_\varphi(A)^2$ and $\gamma(A) = \Phi'(A)^2$. Before Eq (7) can be simplified, some models for β(A) and γ(A) have to be assumed. From Eq. (7), it is evident that the only case when this equation becomes linear is when γ does not depend on A, i.e. the quadratic terms are absent. When this is the case, it is also evident that β'(A) must be a linear function in order to make the equation linear. We assume the following model:

$$\begin{aligned} \beta(A) &= (1/2)\, c^2 A^2 \\ \gamma(A) &= a^2 \end{aligned} \tag{8}$$

where c and a are constants of dimensionality inverse length (c has nothing to do with the velocity of light which has been put to unity). Eq. (7) becomes

$$\begin{aligned} & (1 - a^2 r^2 \sin^2\theta\,)(\partial_r^2 A + \frac{1}{r^2} \partial_\theta^2 A) - 2a^2 r \sin^2\theta\, \partial_r A \\ & - \frac{\cos\theta}{r^2 \sin\theta} \left(1 + a^2 r^2 \sin^2\theta\right) \partial_\theta A + c^2 A = 0. \end{aligned} \tag{9}$$

Unfortunately, this equation is not separable, and thus analytic solution is hardly possible. However, we can calculate from Eq. (9) the asymptotic behaviours when ar » 1 and ar « 1, respectively. When this is done (by the method of separation of variables), one notes that Eq. (9) always has solutions that go to zero as r approaches infinity, and that the magnetic field as well as the scalar potential remain finite at the origin.

IV. CONCLUSION

We have presented an upper energy limit for MHD ball lightning models. We have also introduced the "cold" plasma model for ball lightning and analyzed the equations in certain special cases. It has been found that the equation has solutions that behave asymptotically in the right way. In principle, one can construct "cold" ball lightning models with arbitrarily high energy content. The stability of "cold" models remains to be studied.

ACKNOWLEDGEMENTS

The author is grateful to Dr. Risto Pirjola and Ari Viljanen for various reading the manuscript and making a number of proposals.

1 B.M. Smirnov, Phys. Rep. **152**, 177-226, (1987)

2 H. Wu and Y. Chen, Phys. Fluids B **1**, 1753 (1989)

3 Reference [1], p. 189

4 Reference [1], p. 187